CONCENTRATION OF SYMMETRIC EIGENFUNCTIONS

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ABSTRACT. In this article we examine the concentration and oscillation effects developed by high-frequency eigenfunctions of the Laplace operator in a compact Riemannian manifold. More precisely, we are interested in the structure of the possible invariant semiclassical measures obtained as limits of Wigner measures corresponding to eigenfunctions. These measures describe simultaneously the concentration and oscillation effects developed by a sequence of eigenfunctions. We present some results showing how to obtain invariant semiclassical measures from eigenfunctions with prescribed symmetries. As an application of these results, we give a simple proof of the fact that in a manifold of constant positive sectional curvature, every measure which is invariant by the geodesic flow is an invariant semiclassical measure.

1. INTRODUCTION

The analysis of the concentration and oscillation properties of high-frequency solutions to Schrödinger equations is a central theme in the study of the correspondence principle in quantum mechanics.

Special attention has been devoted to the analysis of the high-frequency behavior of eigenfunctions of the Laplace-Beltrami operator Δ_M of a smooth compact Riemannian manifold (M,g). The spectrum of $-\Delta_M$ consists of a discrete set of eigenvalues (λ_k) tending to infinity. The corresponding eigenfunctions (ψ_{λ_k}):

$$-\Delta_M \psi_{\lambda_k}(x) = \lambda_k \psi_{\lambda_k}(x), \quad x \in M,$$

span the space $L^2(M)$ of square integrable functions with respect to the Riemannian measure dm_g. In this setting, the correspondence principle roughly asserts that high energy eigenfunctions (*i.e.* those corresponding to an eigenvalue λ_k big enough) exhibit behavior that is somehow related to the dynamics of the geodesic flow on (M,g) (see for instance [4, 13, 22] for a more detailed account on this issue).

One associates to any eigenfunction ψ_{λ_k} normalized in $L^2(M)$ the probability distribution $\nu_k := |\psi_{\lambda_k}|^2 \, dm_g$. Then, given any sequence of eigenvalues (λ_k) tending to infinity, one wishes to understand the structure of all possible limits of any sequence (ν_k) (which describe the regions on which ψ_{λ_k} concentrates) and clarify how it is related to the geodesic flow in (M,g). Instead of dealing directly with ν_k it is preferable to associate to ψ_{λ_k} a measure on the cotangent bundle T^*M that projects onto ν_k. These lifts are distributions $W^M_{\psi_{\lambda_k}}$ on T^*M that act on test functions $a \in C_c^\infty(T^*M)$ as:

$$\left\langle W^M_{\psi_{\lambda_k}}, a \right\rangle = \int_M \operatorname{op}_{\lambda_k^{-1/2}}(a)\, \psi_{\lambda_k}(x)\, \overline{\psi_{\lambda_k}(x)} dm_g(x).$$

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In the above formula $\mathrm{op}_h(a)$ stands for the semiclassical pseudodifferential operator of symbol a. There is no canonical for defining $\mathrm{op}_h(a)$, but any two of those differ by a term which vanishes as $h \to 0^+$. When $\mathrm{op}_h(a)$ is given by the Weyl quantization rule, the distribution $W^M_{\psi_{\lambda_k}}$ is usually called the *Wigner measure* of ψ_{λ_k}. More details can be found, for instance, in [5, 6, 7, 15, 22].

Given a sequence of normalized eigenfunctions (ψ_{λ_k}), the corresponding sequence $(W^M_{\psi_{\lambda_k}})$ is bounded in the space of distributions on T^*M, and therefore has at least one accumulation point. When $W^M_{\psi_{\lambda_k}} \rightharpoonup \mu$ as $k \to \infty$ it is by now well known that the limit μ is a Radon probability measure, supported on the unit cosphere bundle S^*M that is invariant by the geodesic flow of (M, g). In addition, the limit of the probability densities ν_k may be recovered by projecting μ onto M. Such a measure μ is usually called an *invariant semiclassical measure*, or a *quantum limit* of (M, g).

The problem of identifying the set of invariant semiclassical measures on a manifold (M, g) has attracted considerable attention in the past thirty years. It turns out that this structure depends heavily on the dynamical properties of the geodesic flow in (M, g). For instance, when it has the Anosov property (which is the case when (M, g) has negative sectional curvature) it is known that a generic invariant semiclassical measure (in a sense to be precised) must coincide with the Liouville measure on S^*M [24, 3, 27, 9, 7, 23]. Moreover, it has been recently proved that quantum limits must have positive Kolmogorov-Sinai entropy [1, 2]. If the geodesic flow is completely integrable, then the invariant semiclassical measures are supported on certain invariant torii of the classical dynamics [12, 13, 25], which in some cases may consist on a single geodesic [14, 17].

In this article, we address the question of whether an invariant semiclassical measure μ may be realized by a sequence (ψ_{λ_k}) of eigenfunctions with specified symmetries, *i.e.* that are invariant by a certain group G of isometries of (M, g). Our main results, Theorem 1 and Corollary 2, give a prodecure to construct invariant semiclassical measures in a Riemannian manifold which is the quotient of M by a group of isometries that act without fixed points. As an application of this, we show in Theorem 3, using essentially only geometric arguments and standard properties of spherical harmonics, that the set of invariant semiclassical measures on a manifold (M, g) of constant, positive sectional curvature coincides with the whole set of probability measures on S^*M that are invariant by the geodesic flow.

Notation and conventions. Through this article (M, g) will denote a smooth, compact, connected Riemannian manifold. We shall denote the geodesic flow on S^*M by ϕ^M_t.

A generic orbit of ϕ^M_t in S^*M will be denoted by γ; when no confusion arises, we shall use the same notation to refer to its projection on M.

Given a group G of isometries of (M, g), we shall use the same notation to refer to the corresponding group of induced diffeomorphisms on S^*M.

In what follows, we shall use the term measure to refer to a probability Radon measure.

2. Main results

Any group G of isometries of (M, g) defines a natural action on the set of ϕ^M_t-invariant measures μ in S^*M by push-forward $\phi_*\mu$, for $\phi \in G$.[1] We denote the

[1] The measure $\phi_*\mu$ is defined by $\phi_*\mu(\Omega) := \mu\left(\phi^{-1}(\Omega)\right)$ for every measurable set $\Omega \subset S^*M$.

stabilizer subgroup of a measure μ with respect to this action by:

$$G_\mu := \{\phi \in G : \phi_* \mu = \mu\}.$$

Recall that a ϕ_t^M-invariant measure μ is ergodic if and only if for every $\Omega \subset S^*M$ which is ϕ_t^M-invariant, $\mu(\Omega)$ must be either zero or one.

Theorem 1. *Let (M, g) be a compact Riemannian manifold and G a finite group of isometries of M. Let μ be a ϕ_t^M-invariant ergodic measure on S^*M that is an invariant semiclassical measure realized by some sequence of G_μ-invariant eigenfunctions. Then*

$$\langle \mu \rangle := \frac{1}{|G|} \sum_{\phi \in G} \phi_* \mu$$

is an invariant semiclassical measure realized by some sequence of G-invariant eigenfunctions.

Theorem 1 can be applied to show the existence of invariant semiclassical measures on quotient Riemannian manifolds. More precisely, if G is a group of isometries that acts without fixed points on M then M/G is a Riemannian manifold in a natural way. Denote the natural projection by

$$\pi : S^*M \to S^*(M/G).$$

The next result shows how invariant semiclassical measures on M/G are constructed from those on M.

Corollary 2. *Let (M, g) be a compact Riemannian manifold, and G be a group of isometries of M that acts without fixed points. Let μ be a ϕ_t^M-invariant, ergodic measure in S^*M that is an invariant semiclassical measure realized by some sequence of G_μ-invariant eigenfunctions in M. Then $\pi_*\mu$ is an invariant semiclassical measure on the quotient Riemannian manifold M/G.*

The proofs of Theorem 1 and Corollary 2 are presented in Section 3. These results may be applied to characterize the set of semiclassical invariant measures on manifolds of positive constant sectional curvature (recall that these manifolds are isometric to quotients of the standard sphere $\mathbb{S}^d$ by a group of isometries acting without fixed points, see [26]). In fact, combining Corollary 2 with rather elementary geometric arguments, and standard properties of spherical harmonics, we shall give in Section 4 a proof of the following theorem.

Theorem 3. *Let (M, g) be a Riemannian manifold of positive constant sectional curvature. Then any ϕ_t^M-invariant measure on S^*M is an invariant semiclassical measure.*

When (M, g) is the standard sphere or the real projective space, Theorem 3 has been proved in [14]. If we further restrict ourselves to the class of homogeneous compact manifolds of constant curvature then it turns out that those with positive curvature are precisely the spaces having the property that the set of invariant semiclassical measures coincides with the whole set of ϕ_t^M-invariant measures. This is due to the fact that there are no such spaces for $K < 0$ and, when $K = 0$, such a space has to be isometric to the flat torus $\mathbb{T}^d$ (see for instance [26]). In the latter case, a result by Bourgain [12] asserts that the projection on $\mathbb{T}^d$ of every semiclassical invariant measure is absolutely continuous with respect to the Lebesgue measure. Therefore, a measure supported on a geodesic cannot be an invariant semiclassical measure.

Corollary 4. *Let (M, g) be a compact homogeneous Riemannian manifold of constant sectional curvature K. All ϕ_t^M-invariant measures on S^*M are invariant semiclassical measures if and only if $K > 0$.*

3. Symmetric eigenfunctions

First of all, let us recall some of the basic properties of Wigner measures. Let (u_k) be a sequence in $L^2(M)$ such that $W_{u_k}^M \rightharpoonup \mu$ as $k \to \infty$ for some measure μ on T^*M. The following properties are well known (see for instance [6, 7]):

$$\text{if } \phi : M \to M \text{ is a diffeomorphism then } W_{u_k \circ \phi}^M \rightharpoonup \phi_* \mu \text{ as } k \to \infty. \tag{1}$$

Let (v_k) be some other sequence such that $W_{v_k}^M \rightharpoonup \nu$ as $k \to \infty$; then

$$\text{if } \mu \perp \nu \text{ then } W_{u_k + v_k}^M \rightharpoonup \mu + \nu \text{ as } k \to \infty. \tag{2}$$

Proof of Theorem 1. Start noticing that given an isometry ϕ, the measure $\phi_*\mu$ is a ϕ_t^M-invariant ergodic measure whenever μ is. Moreover, it is not hard to see, using Birkhoff's ergodic theorem, that either $\phi_*\mu = \mu$ or $\mu \perp \phi_*\mu$.

Indeed, for $\nu \in \{\mu, \phi_*\mu\}$ there exists a measurable set $F_\nu \subset S^*M$ with $\nu(F_\nu) = 1$ and

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T a(\phi_t(x_0, \xi_0))\, dt = \int_{S^*M} a(x, \xi)\, \nu(dx, d\xi),$$

for every $(x_0, \xi_0) \in F_\nu$ and $a \in C(S^*M)$. Therefore, if $F_\mu \cap F_{\phi_*\mu} \neq \emptyset$ then necessarily $\mu = \phi_*\mu$.

Now, the measures $\phi_*\mu$, $\phi \in G$, are pairwise distinct if and only if $G_\mu = \{\mathrm{Id}\}$. In this case, it is easy to construct a sequence of eigenfunctions for which the conclusion holds. Let (ψ_{λ_k}) be such that

$$W_{\psi_{\lambda_k}}^M \rightharpoonup \mu, \qquad \text{as } k \to \infty, \tag{3}$$

and define the average

$$\langle \psi_{\lambda_k} \rangle_G := \frac{1}{|G|} \sum_{\phi \in G} \psi_{\lambda_k} \circ \phi.$$

Clearly, this is a G-invariant eigenfunction of Δ_M (that might possibly vanish identically). Now, because of (1), $W_{\psi_{\lambda_k} \circ \phi}^M$ converges to the measure $\phi_*\mu$, and since all the measures $\phi_*\mu$ are distinct, they must be mutually disjoint. Now, the asymptotic orthogonality property (2) then implies:

$$W_{\langle \psi_{\lambda_k} \rangle_G}^M \rightharpoonup \frac{1}{|G|} \sum_{\phi \in G} \phi_*\mu = \langle \mu \rangle, \qquad \text{as } k \to \infty. \tag{4}$$

Note that, in particular, (4) implies that the sequences of averages $\langle \psi_{\lambda_k} \rangle_G$ is not identically equal to zero, and therefore can be normalized in $L^2(M)$.

Suppose now that G_μ is non-trivial. By hypothesis, there exists (ψ_{λ_k}) such that (3) holds and $\psi_{\lambda_k} \circ \phi = \psi_{\lambda_k}$ for every $\phi \in G_\mu$. Let $\phi_1 = \mathrm{Id}, \phi_2, ..., \phi_{|G|/|G_\mu|}$ be a common system of representatives for the left cosets ϕG_μ and the right cosets $G_\mu \phi$ of G_μ in G (whose existence is ensured by a classical theorem of P. Hall, see for instance [8] Theorem 5.1.7). Given any $\phi \in G$, one has $\psi_{\lambda_k} \circ \rho = \psi_{\lambda_k} \circ \phi$ for every $\rho \in G_\mu \phi$; therefore, the average satisfies:

$$\langle \psi_{\lambda_k} \rangle_G = \frac{|G_\mu|}{|G|} \sum_{l=1}^{|G|/|G_\mu|} \psi_{\lambda_k} \circ \phi_l.$$

Since there is a bijection between the orbit $\{\phi_*\mu : \phi \in G\}$ and the set of left cosets of G_μ in G, all measures $\mu, (\phi_2)_* \mu, ..., (\phi_j)_* \mu$ must be distinct. The conclusion then follows using the same argument we gave for $|G_\mu| = 1$. $\square$

To prove Corollary 2 just take into account the following: (i) the eigenfunctions of $\Delta_{M/G}$ are induced (via π) precisely by the eigenfunctions of Δ_M that are G-invariant; (ii) given any measure μ in M, $\pi_*\mu = \pi_* \langle \mu \rangle$ and

Lemma 5. *It is possible to give a definition of Wigner measures in M and M/G such that for every $u \in L^2(M)$ which is G-invariant, one has that if W_u^M is the Wigner measure of u in M then $W_u^{M/G} = \pi_* W_u^M$ is the Wigner measure of u in M/G.*

The proof of this result follows the classical construction via local charts (see, for instance [16], Section 3); it suffices to construct the Wigner measures from an atlas in (U_i, φ_i), $i = 1, ..r$, in M/G and an atlas $(V_{i,j}, \tilde{\varphi}_{i,j})$ in M such that $V_{i,j} \subset \pi^{-1}(U_i)$, and $\tilde{\varphi}_{i,j} = \pi \circ \varphi_i$, where $\pi : M \to M/G$ is the natural projection. We emphasize the fact that the set of invariant semiclassical measures on a manifold (M, g) does not depend of the notion of Wigner measure used to realize it (see, for instance, [7, 16]).

4. Positive sectional curvature

We now turn to analyze the structure of invariant semiclassical measures in manifolds of constant, positive sectional curvature. Any such manifold (M, g) is the quotient of $\mathbb{S}^d$ by a group G of isometries that acts without fixed points. Recall that the eigenvalues of $-\Delta_{\mathbb{S}^d}$ are $\lambda_k = k(k + d - 1)$ and the corresponding eigenfunctions ψ_k are spherical harmonics of degree k; the eigenfunctions of $-\Delta_M$ are precisely those spherical harmonics that are G-invariant.

Since every ϕ_t^M-invariant measure in S^*M may be approximated by finite convex combinations of measures δ_γ with γ a geodesic (by the Krein-Milman theorem), Theorem 3 is then a consequence of Lemma 5 and of the following result.

Proposition 6. *Let G be a group of order p formed by isometries of $\mathbb{S}^d$ acting without fixed points. Given any geodesic γ on $S^*\mathbb{S}^d$ there exist a sequence (ψ_{kp}) of normalized, G-invariant spherical harmonics such that δ_γ is an invariant semiclassical measure realized by (ψ_{kp}).*

Note that, in particular, this shows that $kp(kp + d - 1)$ are eigenvalues of $-\Delta_M$; more detailed results on the structure of the spectrum of manifolds of constant, positive sectional curvature may be found in [10, 11, 19, 20, 21].

As a consequence of Theorem 1, the proof of Proposition 6 may be reduced to that of the following simpler result.

Proposition 7. *Let G and p be as above. Given any geodesic γ in $S^*\mathbb{S}^d$ there exists a sequence (ψ_{kp}) of normalized G_γ-invariant spherical harmonics such that $W_{\psi_{kp}}^{\mathbb{S}^d} \rightharpoonup \delta_\gamma$ as $k \to \infty$, where G_γ is the subgroup of G consisting of the $\phi \in G$ such that $\phi(\gamma) = \gamma$.*

Note that Proposition 7 is a direct consequence of Theorem 1 in [14] when d is even, since in this case either $G = \{\mathrm{Id}\}$ or $G = \{\mathrm{Id}, -\mathrm{Id}\}$, see [26]. Therefore, we shall assume in what follows that d is odd, and therefore $\mathbb{S}^d$ is contained in an even-dimensional euclidean space $\mathbb{R}^{d+1}$. Write $n := (d+1)/2$, in what follows, we shall identify $\mathbb{R}^{d+1}$ to $\mathbb{C}^n$. The isometries of $\mathbb{S}^d$ that act without fixed points are restrictions to $\mathbb{S}^d$ of maps belonging to $SO(d+1)$. Given any $\phi \in SO(d+1)$, there exist $\varphi \in SO(d+1)$ and $(\theta_1, ..., \theta_n) \in \mathbb{T}^n$ such that

$$\varphi^{-1}\phi\varphi = \begin{bmatrix} e^{i\theta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{i\theta_n} \end{bmatrix}. \tag{5}$$

Our next result clarifies the structure of the groups of isometries that leave a geodesic invariant.

Lemma 8. *Let $H \subset SO(d+1)$ be a finite subgroup that acts without fixed points on $\mathbb{S}^d$. If H leaves a geodesic γ in $\mathbb{S}^d$ invariant then H must be cyclic.*

Proof. Suppose γ is obtained as the intersection of $\mathbb{S}^d$ with a plane $\pi_\gamma \subset \mathbb{R}^{d+1}$ through the origin. Then every $\phi \in H$ leaves invariant both π_γ and $(\pi_\gamma)^\perp$. Therefore, there exists a $\varphi \in SO(d+1)$ such that every $\phi \in G$ is of the form:

$$\phi = \varphi^{-1} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & Q \end{bmatrix} \varphi,$$

for some $\theta \in \mathbb{S}^1$, $Q \in SO(d-1)$. Since ϕ has no fixed points, the order of $e^{i\theta}$ must coincide with the order of ϕ and must divide $p := |H|$. By the same reason, H cannot have elements of the form

$$\varphi^{-1} \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & Q \end{bmatrix} \varphi, \qquad Q \in SO(d-1);$$

unless $Q = \mathrm{Id}$. This shows that H is conjugate in $SO(d+1)$ to a subgroup of the group consisting of the elements

$$\begin{bmatrix} e^{2\pi i j/p} & 0 \\ 0 & Q \end{bmatrix}, \qquad j = 1, ..., p, \quad Q \in SO(d-1),$$

which is isomorphic to $\mathbb{Z}_p \times SO(d-1)$. But any subgroup A of $\mathbb{Z}_p \times SO(d-1)$ having the property that the identity is the only element of the form $(0, Q)$ must necessarily be cyclic.

Indeed, let $C \subset \mathbb{Z}_p$ be the (cyclic) subgroup consisting of the $q \in \mathbb{Z}_p$ such that $(q, h) \in A$ for some $h \in SO(d-1)$. Given any $q \in C$, there exists a unique $h \in SO(d-1)$ such that $(q, h) \in G$ (otherwise, there would exist elements in A of the form $(0, h)$ with $h \neq \mathrm{Id}$); denote it by $\chi(q)$. Clearly, the mapping $\chi : C \to SO(d-1)$ is an injective group homomorphism and A is the graph of χ. If q_0 is a generator of C then necessarily $(q_0, \chi(q_0))$ is a generator of A. □

Proof of Proposition 7. Let $p \in \mathbb{N}$ and take $l_1, ..., l_n$ positive integers less than or equal to p and coprime with p. Denote by $G(p, l_1, ..., l_n)$ the subgroup of $SO(d+1)$ generated by

$$\phi := \begin{bmatrix} e^{2\pi i l_1/p} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{2\pi i l_n/p} \end{bmatrix}; \tag{6}$$

which acts without fixed points. Let γ be a geodesic in $S^*\mathbb{S}^d$ and write $G_\gamma := G_{\delta_\gamma}$. Clearly, G_γ is the subgroup of G consisting of the isometries that leave γ invariant.

It suffices to prove the conclusion for $G_\gamma = G(p, l_1, ..., l_n)$. This is due to the fact that any subgroup G_γ is generated by some element ρ of order $p := |G_\gamma|$, as a consequence of Lemma 8. Now, since ρ is conjugate in $SO(d+1)$ to an element of the form (5) and ρ^k has no fixed points for $1 \le k < p$ we conclude that $G_\gamma = \varphi^{-1} G(p, l_1, ..., l_n)\varphi$ for some $\varphi \in SO(d+1)$ and some positive integers $l_1, ..., l_r \le p$ coprime with p. Let $\tilde{\gamma} := \varphi^{-1}(\gamma)$; the geodesic $\tilde{\gamma}$ clearly satisfies $G_{\tilde{\gamma}} = G(p, l_1, ..., l_n)$; if $\delta_{\tilde{\gamma}}$ is an invariant semiclassical measure realized by a sequence $(\tilde{\psi}_{kp})$ of $G_{\tilde{\gamma}}$-invariant spherical harmonics then $\psi_{kp} := \tilde{\psi}_{kp} \circ \varphi$ is a sequence of G_γ-invariant spherical harmonics satisfying (because of (1)) $W^{\mathbb{S}^d}_{\psi_{kp}} \rightharpoonup \delta_{\varphi(\tilde{\gamma})} = \delta_\gamma$ as $k \to \infty$.

The conclusion holds for $G_\gamma = G(p, l_1, ..., l_n)$. Start considering the geodesics γ_j defined by $|x_{2j-1}|^2 + |x_{2j}|^2 = 1$. Let

$$\psi_k^0(x) := C_k (x_{2j-1} + i x_{2j})^k ,$$

where $C_k > 0$ is chosen to have $\left\|\psi_k^0\right\|_{L^2(\mathbb{S}^d)} = 1$. Clearly, ψ_k^0 is a spherical harmonic of degree k and, as is well known (see for instance [14]), $W^{\mathbb{S}^d}_{\psi_k^0} \rightharpoonup \delta_{\gamma_j}$ as $k \to \infty$. Moreover, it easy to check that

$$\psi_k^0(\phi(x)) = C_k e^{2\pi i k l_j/p} (x_{2j-1} + i x_{2j})^k ;$$

and in particular $\psi_{kp}^0 \circ \phi = \psi_{kp}^0$. Therefore δ_{γ_j} is a G_γ-invariant semiclassical measure realized by the sequence (ψ_{kp}^0).

Let now γ be a ϕ-invariant geodesic in $\mathbb{S}^d$. Suppose that there exists a $\chi \in SO(d+1)$, commuting with ϕ, and such that $\chi(\gamma_j) = \gamma$; clearly $\psi_{kp} := \psi_{kp}^0 \circ \chi$ is again ϕ-invariant and δ_γ is an invariant semiclassical measure realized by (ψ_{kp}). The proof will be concluded as soon as we show that such a χ exists. If the geodesic γ differs from the γ_j then it must be the intersection of $\mathbb{S}^d$ with a plane $\pi_\gamma \subset \mathbb{R}^{d+1}$ which is also a complex line in $\mathbb{C}^n$ and that is invariant by ϕ. Therefore, it must be contained in a linear subspace E_γ of $\mathbb{C}^n$ on which ϕ acts as multiplication by some fixed $e^{2\pi i l_j/p}$. Define χ as an element of $SU(n) \subset SO(d+1)$ such that $\chi(\gamma_j) = \gamma$ and χ is the identity on the orthogonal of E_γ. Clearly, $\chi_{|E_\gamma}$ commutes with multiplication by $e^{2\pi i l_j/p}$ and the result follows. □

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